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Common Fixed Point Theorems for Four Selfmaps of a Compact S−Metric Space

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ABSTRACT

The purpose of this paper is to prove a common fixed point theorem for four selfmaps on a S–metric space and deduce a common fixed point theorem for four selfmaps on a compact S–metric space. Further we show that a common fixed point theorem for four selfmaps of a metric space prove by Brian Fisher ([5]) is a particular case of our theorem.

Mathematics Subject Classification: 47H10, 54H25.

Keywords: *S-metric space; Compatible; Fixed point theorem.*

1. INTRODUCTION AND PRELIMINARIES:

Fixed point theory is an important discipline in mathematics because of its results which are utilized to investigate the existence of solutions for the problems in applied sciences and engineering. Many fixed point results have been widely generalized throughout the years in various directions by introducing new metric spaces and setting of new contraction mappings. The results in fixed point theory can be noticed in geometry, computational algorithms, economics, fluid dynamics, micro-structures, nonlinear sciences, medical fields and optimization theory.

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [6] gave the concepts of 2- metric space further in 1992 Dhage [2] modified the concept of 2- metric space and introduced the concepts of D-metric space also proved fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [7], [8] and [9]. Sedghi et al. [10] modified the concepts of D- metric space and introduced the concepts of D*- metric space also proved a common fixed point theorems in D*- metric space.

Recently, Sedghi et al [11] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different from other spaces. In fact they give following concepts of S- metric space.

Definition 1.1([11]): Let X be a non-empty set. An S-metric space on X is a function S: $X^3 \rightarrow [0, \infty)$ that satisfies the following conditions, for each x, y, z, a ∈ X

(i) $S(x, y, z) \geq 0$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$.

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an **S–metric space**. Immediate examples of such S-metric spaces are:

Example1.2: Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - y| + |y - z| + |z - x|$ for each $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Example 1.3: Let $X = \mathbb{R}^2$, d be the ordinary metric on X.

Put $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is an S- metric on X. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ holds. In fact

$$S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$
$$\leq d(x, a) + d(a, y) + d(y, a) + d(a, z) + d(z, a) + d(a, x)$$
$$= S(x, x, a) + S(y, y, a) + S(z, z, a)$$

Example1. 4: Let $X = \mathbb{R}^n$ and $\|\ .\ \|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Remark1. 5: it is easy to see that every D*-metric is S-metric, but in general the converse is not true, see the following example.

Example1. 6: Let $X = \mathbb{R}^n$ and $\|\ .\ \|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X, but it is not D*-metric because it is not symmetric.

Lemma1. 7: In an S–metric space, we have $S(x, x, y) = S(y, y, x)$.

Proof: By the third condition of S-metric, we get

$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x)$…… (1)

and similarly

$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y)$……(2)

Hence, by (1) and (2), we obtain $S(x, x, y) = S(y, y, x)$.

Definition1.8: Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r as follows

$$B_S(x, r) = \{y \in X;\ S(x, y, y) < r\}$$
$$B_S[x, r] = \{y \in X;\ S(x, y, y) \leq r\}$$

For example, Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$.

Therefore $B_S(1, 2) = \{y \in \mathbb{R} ; S(y, y, 1) < 2\}$

$= \{y \in \mathbb{R} ; | y - 1| < 1\} = (0, 2)$.

Definition 1.9: Let (X, S) be an S–metric space and $A \subset X$.

(1)If for every $x \in A$, there is a $r > 0$ such that $B_S(x, r) \subset A$, then the subset A called an **open subset** of X

(2) If there is a $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$ then A is said to be **S–bounded.**

(3) A sequence $\{x_n\}$ in X **converge to x** if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for

each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \in$ and we denote this by $\lim_{n \to \infty} x_n = x$

(4) A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \in$ for each $m , n \geq n_0$

(5) The S–metric space (X, S) is said to be **complete** if every Cauchy sequence is convergent sequence.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

(7) If (X, τ) is a compact topological space we shall call (X, S) is a **compact** S–metric space.

Lemma1. 10([11]): Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the open ball $B_S(x, r)$ is an open subset of X.

Lemma1. 11([11]**):** Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma1. 12([11]**):** Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma1. 13([11]**):** Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such

that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$,

then $\lim_{n \to \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma1. 14: Let (X, d) be a metric space. Then we have

1. $S_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$ is an S-metric on X

2. $x_n \to x$ in (X, d) if and only if $X_n \to x$ in (X, S_d)

3. $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d)

4. (X, d) is complete if and only if (X, S_d) is complete

Proof: **(1)** See [Example (3), Page 260]

(2) $x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$, if and only if $S_d(x_n, x_n, x) = 3d(x_n, x) \to 0$ that is, $x_n \to x$ in (X, S_d)

(3) $\{x_n\}$is a Cauchy in (X, d) if and only if $d(x_n, x_m) \to 0$ as n, m $\to \infty$, if and only if $S_d(x_n, x_n, x_m) = 3d(x_n, x_m) \to 0$ n, m $\to \infty$, that is, $\{x_n\}$ is Cauchy in (X, S_d)

(4) It is a direct consequence of (2) and (3)

Notation: For any selfmap T of X, we denote T(x) by Tx.

If P and Q are selfmaps of a set X, then any z ∈ X such that Pz = Qz = z is called a **common fixed point** of P and Q.

Two selfmaps P and Q of X are said to be **commutative** if PQ = QP where PQ is their composition PoQ defined by (PoQ) x = PQx for all x ∈ X.

Definition 1.15: Suppose P and Q are selfmaps of a S–metric space (X, S) satisfying the condition Q(X) ⊆ P(X). Then for any x_0 ∈ X, Qx_0 ∈ Q(X) and hence Qx_0 ∈ P(X), so that there is a x_1 ∈ X with $Qx_0 = Px_1$, since Q(X) ⊆ P(X). Now Qx_1 ∈ Q(X) and hence there is a $x_2 \in X$ with Qx_2 ∈ Q(X) ⊆ P(X) so that $Qx_1 = Px_2$. Again Qx_2 ∈ Q(X) and hence Qx_2 ∈ P(X) with $Qx_2 = Px_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Qx_n = Px_{n+1}$ for n ≥ 0. We shall call this sequence as an **associated sequence of x_0 relative to the two selfmaps P and Q.** It may be noted that there may be more than one associated sequence for a point x_0 ∈ X relative to selfmaps P and Q.

Let P and Q are selfmaps of a S-metric space (X, S) such that Q(X) ⊆ P(X). For any x_0 ϵ X, if $\{x_n\}$ is a sequence in X such that $Qx_n = Px_{n+1}$ for n ≥ 0, then $\{x_n\}$ is called an **associated sequence** of x_0 relative to the two selfmaps P and Q.

Definition 1.16: A function Ø: [0, ∞) ⟶ [0, ∞) is said to be a **contractive modulus**, if Ø (0) = 0 and Ø (t) < t for t > 0.

Definition 1.17: A real valued function Ø defined on X ⊆ ℝ is said to be **upper semi continuous**, if $\limsup_{n\to\infty} Ø(t_n) \le Ø(t)$ for every sequence $\{t_n\}$ in X with t_n ⟶ t as n ⟶ ∞.

Definition 1.18: If P and Q are selfmaps of a S-metric space (X, S) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n$ = t, we have

$\lim_{n\to\infty} S(PQx_n, QPx_n, QPx_n) = 0$, then we say that P and Q are **compatible.**

THE MAIN RESULTS:

2.1 Theorem. Suppose P, T, I and J be selfmaps of a S–metric space (X, S) satisfying the conditions

(i) $P(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$

(ii) $S(Px, Ty, Ty) \leq \rho(x, y)$ for all $x, y \in X$,

where

(ii)′ $\rho(x, y) = \max\{S(Ix, Jy, Jy), S(Ix, Px, Px), S(Jy, Ty, Ty), \frac{1}{2} S(Ix, Ty, Ty), \frac{1}{2} S(Jy, Px, Px)\}$ for $x, y \in X$

(iii) P, T, I and J are continuous.

(iv) the pairs (P, I) and (T, J) are compatible, and

(v) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the four selfmaps such that the sequences $\{Px_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to some point $z \in X$

Further, if

(vi) there exists $(a, b) \in X^2$ such that $f(a, b) = \sup_{(x,y)\in X^2} f(x, y)$,

where

(vi)′ $f(x, y) = \frac{S(Px, Ty, Ty)}{\rho(x,y)}$

then P, T, I and J have a unique common fixed point $z \in X$. also z is the unique fixed point for the pair (P, I) and for the pair (T, J).

Proof: First suppose that $\rho(x', y') = 0$ for some $x', y' \in X$. Then

(2. 1. 1) $\max\{S(Ix', Jy', Jy'), S(Ix', Px', Px'), S(Jy', Ty', Ty'), \frac{1}{2} S(Ix', Ty', Ty'), \frac{1}{2} S(Jy', Px', Px')\} = 0$,

which implies

(2.1.2) $Ix' = Px' = Jy' = Ty'$, and also

(2. 1. 3) $PIx' = P(Px') = P^2x'$ and

(2. 1. 4) $TJy' = T(Ty') = T^2y'$. Now since the pair (P, I) is compatible, we have

(2.1.5) $\lim_{n\to\infty} S(PIy_n, IPy_n, IPy_n) = 0$

whenever $Py_n, Iy_n \to t$ as $n \to \infty$ for some $t \in X$. Letting $y_n = x'$ for $n \geq 1$, then $Py_n \to Px'$ and $Iy_n \to Ix'$ as $n \to \infty$. Therefore (2.1.5) gives that $S(PIx', IPx', IPx') = 0$ or $PIx' = IPx'$. Also since $IPx' = P^2x' = PIx'$ and $Jy' = Ty'$ we get

$\rho(Px', y') = \max\{S(IPx', Jy', Jy'), S(IPx', P^2x', P^2x'), S(Jy', Ty', Ty'), \frac{1}{2} S(IPx', Ty', Ty'), \frac{1}{2} S(Jy', P^2x', P^2x')\}$

$= \max\{S(P^2x', Ty', Ty'), 0, 0, \frac{1}{2} S(P^2x', Ty', Ty'), \frac{1}{2} S(P^2x', Ty', Ty')\}$

$= S(P^2x', Ty', Ty')$. That is

(2. 1. 6) $\rho(Px', y') = S(P^2x', Ty', Ty')$

Now if $Ty' \neq P^2x'$, then by (ii), we have

(2.1.7) $S(P^2x', Ty', Ty') < \rho(Px', y')$

Thus (2.1.6) and (2.1.7) contradict each other if $Ty' \neq P^2x'$. Therefore $P^2x' = Ty'$. Further, from (2. 1. 2)

(2. 1. 8) $P^2x' = Ty' = P(Px') = PTy'$ and so $Ty' = z$(say) is a fixed point of P. Again, by (2. 1. 2)

(2. 1. 9) $Iz = ITy' = IPx' = PTy' = Pz = z$. Therefore $Pz = Iz = z$, showing that z is a common fixed point of P and I. Again since the pair (T, J) is compatible, we have $\lim_{n\to\infty} S(TJy_n, JTy_n, JTy_n) = 0$. Whenever Ty_n, $Jy_n \to t$ as $n \to \infty$ for some $t \in X$. Taking $y_n = y'$, we find that $Ty_n \to Ty'$, $Jy_n \to Jy'$ as $n \to \infty$. Therefore (2.1.5) gives that

(2. 1. 10) $S(TJy', JTy', JTy') = 0$ or $TJy' = JTy'$.

Now if $Px' \neq T^2y'$, then by (ii), we have

(2.1.11) $S(Px', T^2y', T^2y') < \rho(x', Ty')$

But, by (2. 1. 2) and (2. 1. 4)we have

$\rho(x', Ty') = \max\{S(Ix', JTy', JTy'), S(Ix', Px', Px'), S(JTy', T^2y', T^2y'), \frac{1}{2} S(Ix', T^2y', T^2y'), \frac{1}{2} S(JTy', Px', Px')\}$

$= S(Px', T^2y', T^2y')$. That is,

(2. 1. 12) $\rho(x', Ty') = S(Px', T^2y', T^2y')$

Thus (2. 1. 11) and (2. 1. 12) contradict each other if $Px' \neq T^2y'$.

Therefore $Px' = T^2y'$. Hence, by (2. 1.10) and (2. 1. 2), we have

(2. 1. 13) $Px' = T^2y' = T(Ty') = TJy' = JTy' = JPx'$, showing that $Px' = z'$ is a fixed point of J. Further

(2. 1. 14) $Tz' = TPx' = TJy' = JTy' = JPx' = Jz' = z'$ and therefore $Tz' = Jz' = z'$, showing that z' is a common fixed point of T and J.

Now we prove that $z = z'$.

First note that, if $z \neq z'$, then by (ii), we have

(2. 1. 15) $S(z, z', z') = S(Pz, Tz', Tz') < \rho(z, z')$. But

(2. 1. 16) $\rho(z, z') = \max\{S(Iz, Jz', Jz'), S(Iz, Pz, Pz), S(Jz', Tz', Tz'), \frac{1}{2} S(Iz, Tz', Tz'), \frac{1}{2} S(Jz', Pz, Pz)\} = 0$

$= \max\{S(z, z', z'), 0, 0, \frac{1}{2} S(z, z', z'), \frac{1}{2} S(z, z', z')\}$

$= S(z, z', z')$,

Since (2. 1. 15) and (2. 1. 16) contradict each other if $z \neq z'$, it follows that z = z'. Hence z is the unique common fixed point of P, T, I and J.

Now suppose that ρ(x, y) > 0 for all x, y ∈ X, so that f(x, y) is well defined. Now by the inequality (ii), we find that f(x, y) < 1 for all x, y ∈ X. Hence if c = f(p, q) then c ≤ 1, so that f(x, y) ≤ c for all x, y ∈ X and therefore, from (vi)′ S(Px, Ty, Ty) ≤ c ρ(x , y) for all x, y ∈ X

Since, by hypothesis, all the conditions of the corollary holds for the four selfmaps P, T, I and J; it follows that they have a common fixed point z ∈ X. Further z is the unique common fixed point of P and I; and of T and J.

To prove the uniqueness of z, let w be another common fixed point of P, T, I and J. If w ≠ z, then by (ii), we have

(2. 1. 17) S(z, w, w) = S(Pz, Tw, Tw) < ρ(z, w)

(2. 1. 18) ρ (z,, w) = max {S(Iz, Jw, Jw), S(Iz, Pz, Pz), S(Jw, Tw, Tw), $\frac{1}{2}$ S(Iz, Tw, Tw), $\frac{1}{2}$ S(Jw, Pz, Pz)}

= max {S(z, w, w), 0, 0, $\frac{1}{2}$ S(z, w, w), $\frac{1}{2}$ S(z, w, w)}

= S(z, w, w),

Now (2. 1. 17) and (2. 1. 18) contradict each other if z ≠ w. Therefore z = w, showing z is the unique common fixed point of P, T, I and J. Further z is the unique common fixed point of P and I; and of T and J.

Now we prove some consequences of Theorem 2. 1

2.2 Corollary: Suppose (X, S) is a S-metric space and P, T, I and J are selfmaps of X satisfying conditions (i), (ii), (iii) and (iv) of Theorem 2.1. Further, if (X, S) is compact, then P, T, I and J have a unique common fixed point z. Also z is the unique common fixed point for the pair P and I; and for the pair T and J.

Proof: Since (X, S) is a compact S-metric space, it is complete and therefore for each x_0 ∈ X and for any associated sequence $\{x_n\}$ of x_0 relative to four selfmaps such that the sequences $\{Px_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to some z ∈ X and hence condition (v) of Theorem 2.1 holds . Also, if (X, S) is compact S-metric space, then f(x, y) is a continuous function on the compact S-metric space X^2. Therefore we can find (a, b) ∈ X^2 such that f(a, b) = $\sup_{(x,y)\in X^2} f(x,y)$, proving that the condition (vi) of the Theorem 2.1. Hence by Theorem 2. 1, the conclusion of the corollary follows.

2.3 Corollary ([5]): Suppose P, T, I and J are four selfmaps of metric space (X, d) such that

(i) $P(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$

(ii) $d(Px, Ty) < \rho_0(x, y)$ for all $x, y \in X$.

where

$\rho_0(x, y) = \max\{d(Ix, Jy), d(Ix, Px), d(Jy, Ty), \frac{1}{2} d(Ix, Ty), \frac{1}{2} d(Jy, Px)\}$

(iii) P, T, I and J are continuous on X. and

(iv) PI=IP and TJ = JT, further if

(v) X is compact.

Then the four selfmaps P, T, I and J have a unique common fixed point $z \in X$. Also z is the unique common fixed point of P and I; and of T and J.

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary. If $S(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, then (X, S) is a S-metric space and $S(x, y, x) = d(x, y)$. Therefore (ii) can be written as $S(Px, Ty, Ty) < \rho(x, y)$ for all $x, y \in X$, where $\rho(x, y) = \max\{S(Ix, Jy, Jy), S(Ix, Px, Px), S(Jy, Ty, Ty), \frac{1}{2} S(Ix, Ty, Ty), \frac{1}{2} S(Py, Tx, Tx)\}$, which is the same as condition (ii) of Theorem 2.1. Also since (X, d) is complete, we have (X, S) is complete, by Corollary 1.14. Now P and T are selfmaps on (X, S) satisfying conditions of Corollary 2.2 and hence the corollary follows.

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